low-Q second-order circuits

It is reasonable to attempt to build second-order circuits using two first-order circuits.

$$T_{lp} = G_o \cdot \frac{\omega_c}{s + \omega_c} \qquad T_{hp} = G_o \cdot \frac{s}{s + \omega_c}$$

$$T_{lp} = \left[G_{o1} \cdot \frac{\omega_{c1}}{s + \omega_{c1}} \right] \cdot \left[G_{o2} \cdot \frac{\omega_{c2}}{s + \omega_{c2}} \right]$$

$$= \left(G_{o1} G_{o2} \right) \frac{\omega_{c1}\omega_{c2}}{s^2 + \left(\omega_{c1} + \omega_{c2} \right) s + \omega_{c1}\omega_{c2}}$$

$$= \left(G_{oT} \right) \frac{\omega_o^2}{s^2 + \left(\frac{\omega_o}{Q_P} \right) s + \omega_o^2}$$

$$G_{oT} = G_{o1} G_{o2} \qquad \omega_o^2 = \omega_{c1}\omega_{c2} \qquad Q_P = \frac{\sqrt{\omega_{c1}\omega_{c2}}}{\omega_{c1} + \omega_{c2}}$$

$$T_{hp} = \left[G_{o1} \cdot \frac{s}{s + \omega_{c1}} \right] \cdot \left[G_{o2} \cdot \frac{s}{s + \omega_{c2}} \right]$$

$$= \left(G_{o1} G_{o2} \right) \frac{s^2}{s^2 + s \left(\omega_{c1} + \omega_{c2} \right) + \omega_{c1} \omega_{c2}}$$

$$= \left(G_{oT} \right) \frac{s^2}{s^2 + \left(\frac{\omega_o}{Q_P} \right) s + \omega_o^2}$$

$$\omega_o^2 = \omega_{c1} \omega_{c2}$$

$$G_{oT} = G_{o1} G_{o2}$$

$$Q_P = \frac{\sqrt{\omega_{c1} \omega_{c2}}}{\omega_{c1} + \omega_{c2}}$$

$$T_{bp} = \left[G_{o1} \cdot \frac{\omega_{c1}}{s + \omega_{c1}}\right] \cdot \left[G_{o2} \cdot \frac{s}{s + \omega_{c2}}\right]$$

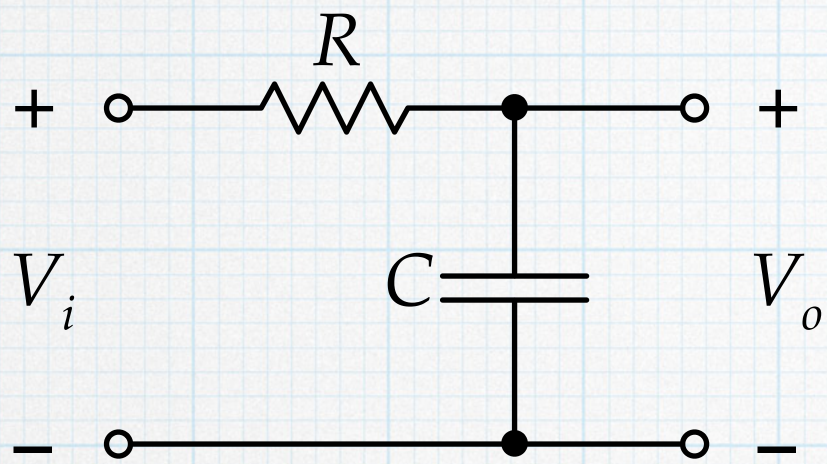
$$= \left(G_{o1}G_{o2}\right) \frac{\omega_{c1}s}{s^2 + s\left(\omega_{c1} + \omega_{c2}\right) + \omega_{c1}\omega_{c2}}$$

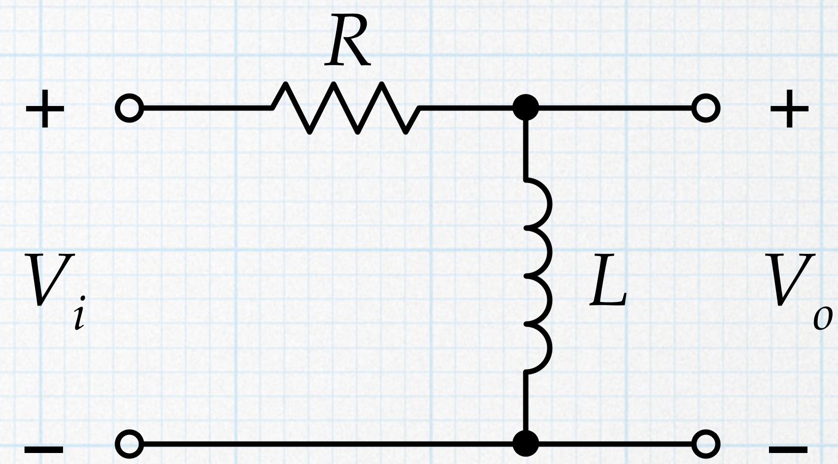
$$= G_{oT} \cdot \frac{\left(\frac{\omega_o}{Q_P}\right)s}{s^2 + \left(\frac{\omega_o}{Q_P}\right)s + \omega_o^2}$$

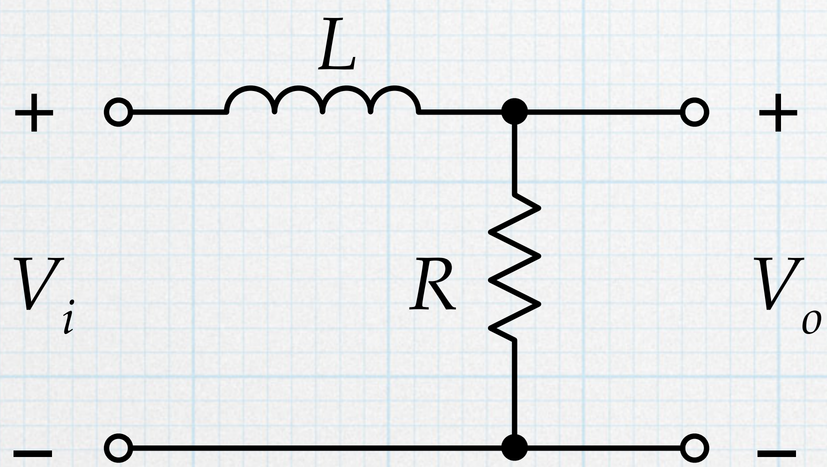
$$\omega_o^2 = \omega_{c1}\omega_{c2}$$

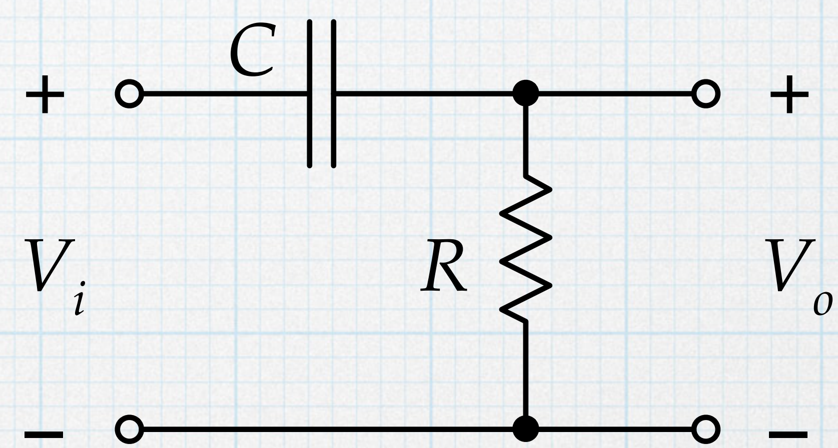
$$G_{oT} = G_{o1}G_{o2}\frac{\omega_{c1}}{\omega_{c1} + \omega_{c2}}$$

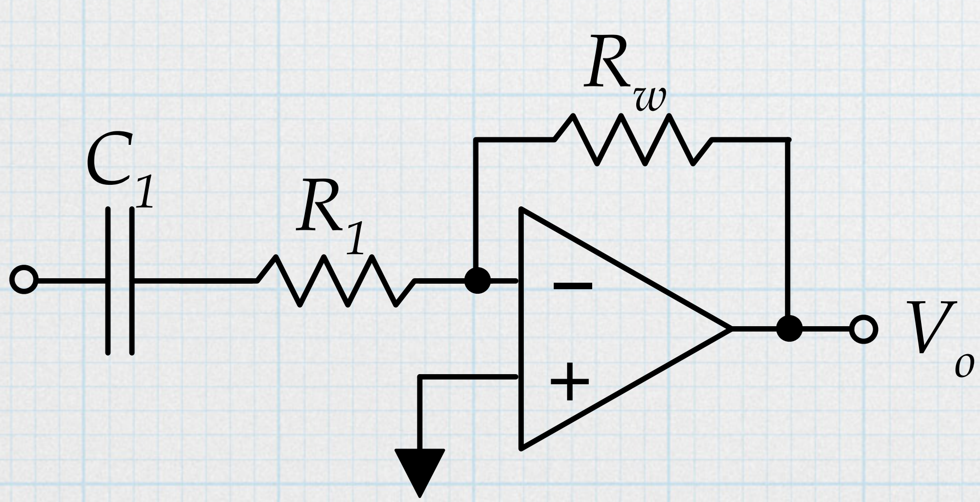
$$Q_P = \frac{\sqrt{\omega_{c1}\omega_{c2}}}{\omega_{c1} + \omega_{c2}}$$

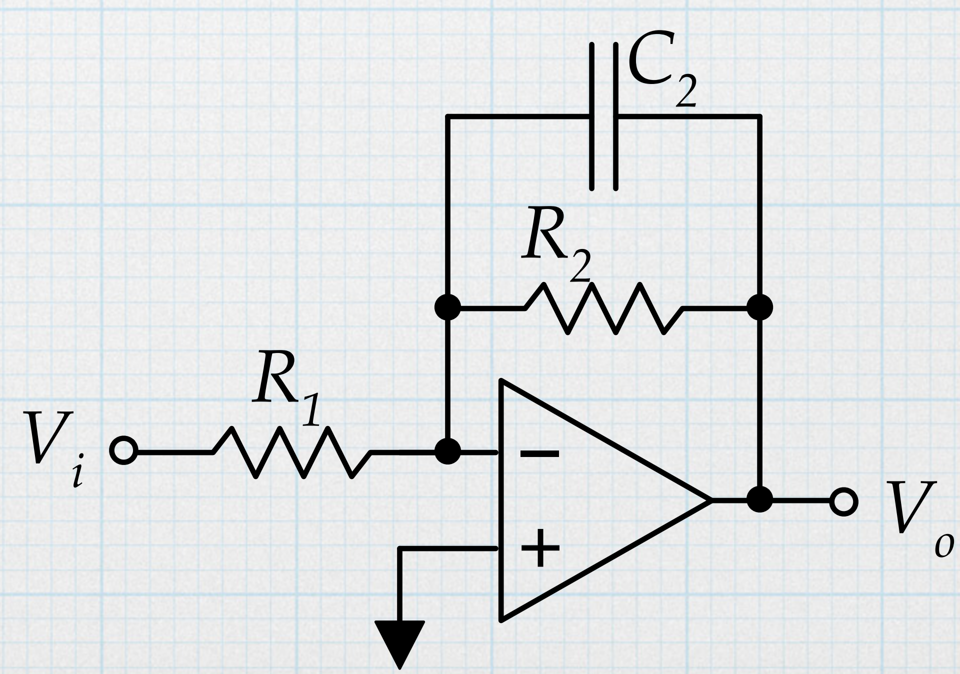

R
C
+
–
+
–
V_i
V_o


R
L
+
–
+
–
V_i
V_o


L
R
+
–
+
–
V_i
V_o


C
R
+
–
+
–
V_i
V_o


C_1
R_1
R_w
–
+
V_o

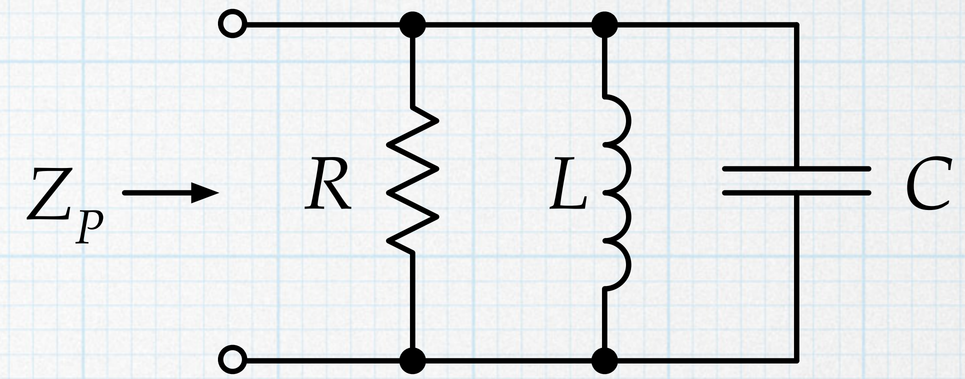

C_2
R_2
R_1
V_i
–
+
V_o

As noted earlier, in order to achieve filters with higher Q_P (> 0.5), some sort of resonance effect is required.

The simplest approach with passive circuits is to use capacitors and inductors together to create the energy-trading, back-and-forth tug-of-war that occurs with inductors and capacitors together. At the resonance frequency, the AC impedances of the two components effect cancel, leaving behind only the resistive part.

As we saw in EE 201, the fact the impedances cancel does not mean that they are gone. In fact the sinusoidal voltages across the reactive components can have a very big magnitude — even bigger than the source. This is not some violation of basic physics. The extra voltage (and hence extra energy) were built up during the transient time that the sinusoids were evolving towards their steady-state values. Since AC analysis intentionally ignores the initial transient, the large resonance voltages (or currents) seem to have come out of nowhere. If we try to use the "extra" energy, we would see a transient flow of energy out of the reactive components, matching the transient that built up the energy in the first place.

To synthesize passive versions the different second-order filter functions consider the simple *RLC* parallel combination shown. We know that this circuit will exhibit resonance.



The impedance of the parallel combination is:

$$\frac{1}{Z_P} = \frac{1}{R} + \frac{1}{sL} + sC$$

After a bit of algebraic finagling:

$$Z_P = \frac{s^2\left(\frac{1}{sC}\right)}{s^2 + s\left(\frac{1}{RC}\right) + \frac{1}{LC}} = \frac{Z_C \cdot s^2}{s^2 + s\left(\frac{1}{RC}\right) + \frac{1}{LC}}$$

We recognize this as the impedance of a capacitor multiplied by a high-pass function.

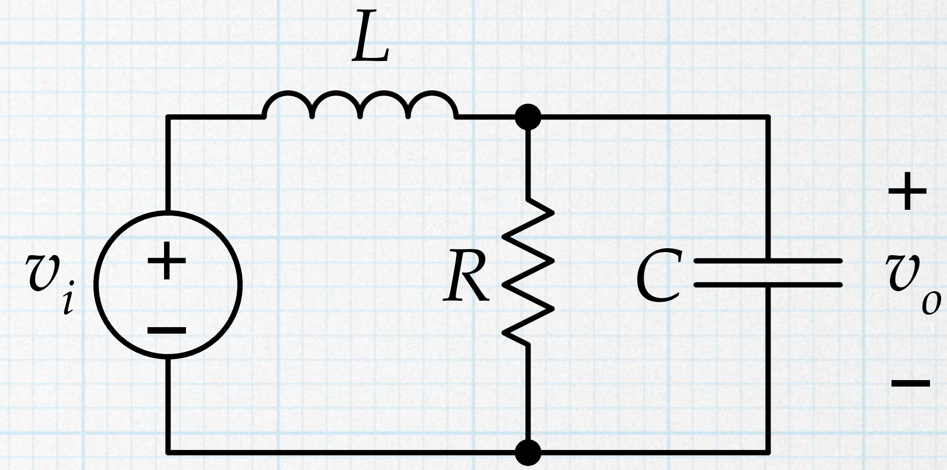
$$Z_P = Z_C \cdot \frac{s^2}{s^2 + s\left(\frac{\omega_o}{Q_P}\right) + \omega_o^2}$$

$$\omega_o^2 = \frac{1}{LC}$$ Resonance frequency of the circuit.

$$Q_P = \omega_o RC$$

Low-pass

Given the characteristics of the parallel combination, it seems likely that second-order transfer functions could be realized using it as a starting point. Consider the circuit at right. It can be viewed as a simple voltage divider with Z_L and $Z_P = Z_R||Z_C$.



$$Z_P = R \left\| \left(\frac{1}{sC} \right) = \frac{R}{1 + sRC} \right.$$

$$V_o(s) = \frac{Z_P}{Z_P + Z_L} V_i(s)$$

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_P}{Z_P + Z_L} = \frac{\frac{R}{1 + sRC}}{\frac{R}{1 + sRC} + sL}$$

After some algebraic jujitsu:

$$T(s) = \frac{\frac{1}{LC}}{s^2 + s\left(\frac{1}{RC}\right) + \frac{1}{LC}} = \frac{\omega_o^2}{s^2 + s\left(\frac{\omega_o}{Q_P}\right) + \omega_o^2}$$

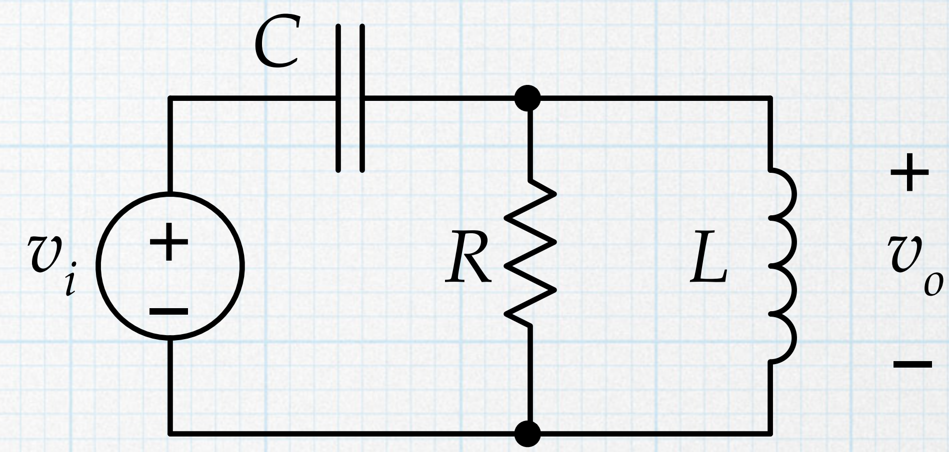
$$G_o = 1$$

$$\omega_o^2 = \frac{1}{LC}$$

$$Q_P = \omega_o RC$$

High-pass

It is probably not surprising that swapping the capacitance and inductor in the low-pass circuit changes the transfer function to a high-pass. Again, the circuit is a simple voltage divider with Z_C and $Z_P = Z_R||Z_L$.



$$Z_P = R\|sL = \frac{sLR}{R + sL}$$

Jumping directly to the transfer function:

$$T(s) = \frac{Z_P}{Z_P + Z_C} = \frac{\frac{sLR}{R+sL}}{\frac{sLR}{R+sL} + \frac{1}{sC}}$$

After some algebraic gymnastics:

$$T(s) = \frac{s^2}{s^2 + s\left(\frac{1}{RC}\right) + \frac{1}{LC}} = \frac{s^2}{s^2 + s\left(\frac{\omega_o}{Q_P}\right) + \omega_o^2}$$

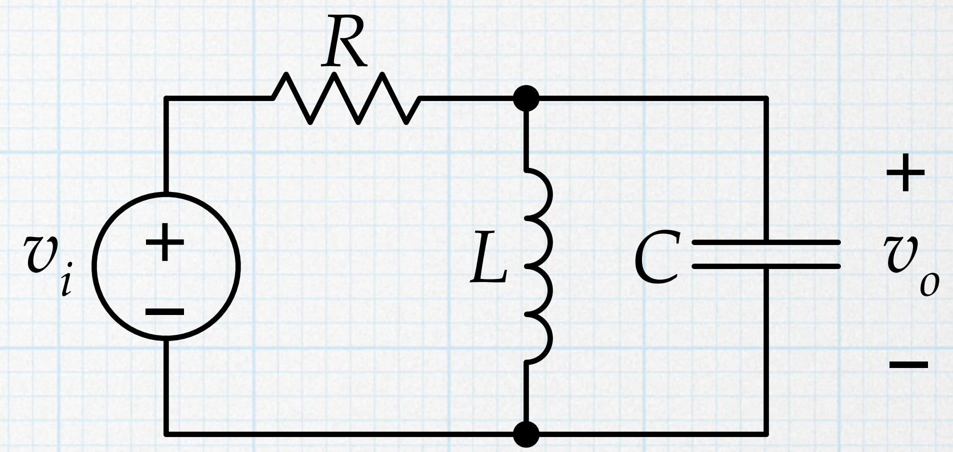
$$G_o = 1$$

$$\omega_o^2 = \frac{1}{LC}$$

$$Q_P = \omega_o RC$$

Band-pass

One more variation of the basic circuit — moving the resistor so that it forms a voltage divider with the inductor/capacitor pair — gives a band-pass response. The divider uses Z_R and $Z_P = Z_L||Z_C$.



$$Z_P = sL \left\| \left(\frac{1}{sC} \right) \right. = \frac{sL}{1 + s^2LC}$$

The transfer function is:

$$T(s) = \frac{Z_P}{Z_P + Z_R} = \frac{\frac{sL}{1 + s^2LC}}{\frac{sL}{1 + s^2LC} + R}$$

After some algebraic sleight-of-hand:

$$T(s) = \frac{s\left(\frac{1}{RC}\right)}{s^2 + s\left(\frac{1}{RC}\right) + \frac{1}{LC}} = \frac{s\left(\frac{\omega_o}{Q_P}\right)}{s^2 + s\left(\frac{\omega_o}{Q_P}\right) + \omega_o^2}$$

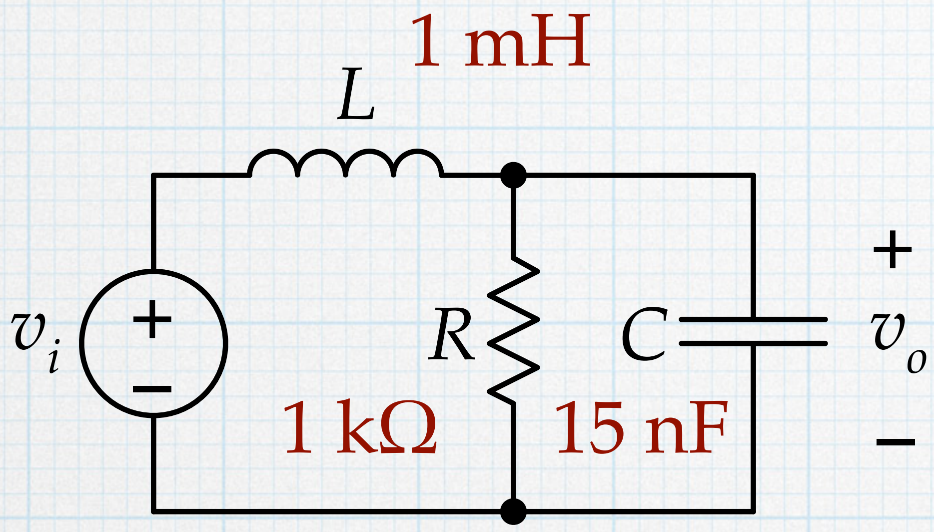
$$G_o = 1$$

$$\omega_o^2 = \frac{1}{LC}$$

$$Q_P = \omega_o RC$$

Example

What type of filter is this? What are ω_o, Q_P, and G_o for this filter? What value of resistance would be needed to make $Q_P = 1$?



From the configuration, we see that this is a low-pass filter.

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1\ \text{mH})\,(15\ \text{nF})}} = 25,800\ \text{rad/s}$$

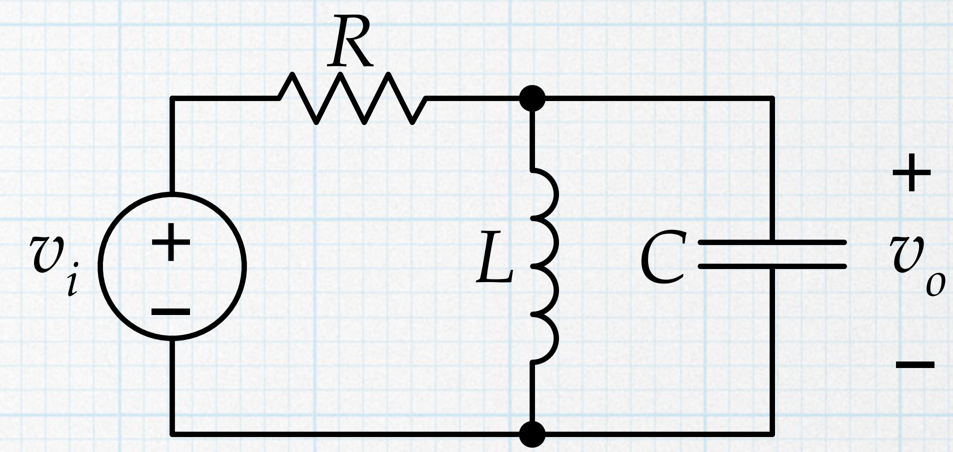
$$Q_P = \omega_o RC = (25,800\ \text{rad/s})\,(1\ \text{k}\Omega)\,(15\ \text{nF}) = 0.4275$$

$$G_o = 1.$$

To make $Q_P = 1$, R would have to be increased to 2.34 kΩ.

Example

Design the band-pass circuit at right to have a center frequency at 3 kHz and a bandwidth of 300 Hz.



$$\omega_o = 2\pi f_o = 2\pi\left(3000 Hz\right) = 18.85 \text{ krad/s}$$

Choose C = 220 nF (somewhat arbitrarily). Then

$$L = \frac{1}{\omega_o^2 C} = \frac{1}{\left(18.85 \text{ krad/s}\right)^2 \left(220 \text{ nF}\right)} = 12.8 \text{ mH}$$

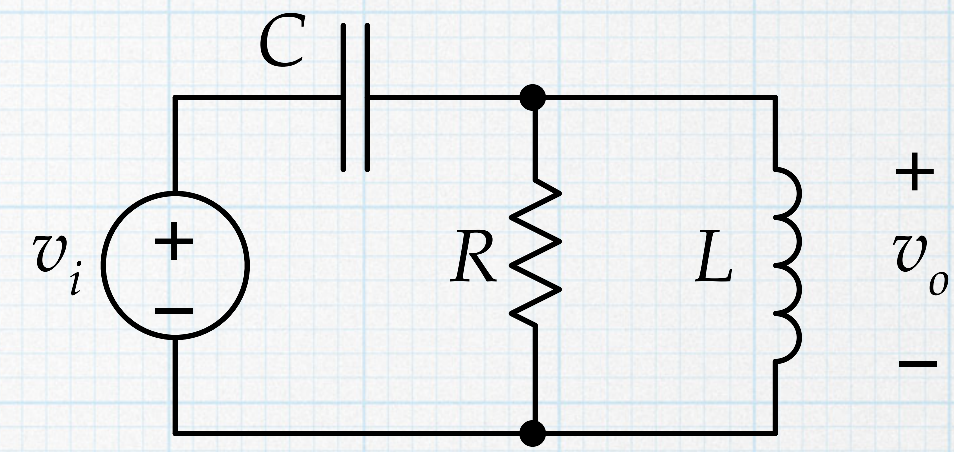
$$\text{BW} = \frac{\omega_o}{Q_P} \qquad Q_P = \frac{\omega_o}{\text{BW}} = 10$$

$$\frac{\omega_o}{Q_P} = \frac{1}{RC} \qquad R = \frac{Q_P}{\omega_o C} = \frac{10}{\left(18.85 \text{ krad/s}\right)\left(220 \text{ nF}\right)} = 2.41 \text{ k}\Omega$$

(Might need to fiddle around with values a bit to find the right components.)

Example

Design a high-pass filter, as shown at right, to be "maximally" flat with corner frequency at 500 Hz.



Maximally flat means that $Q_P = \frac{1}{\sqrt{2}} = 0.7071$

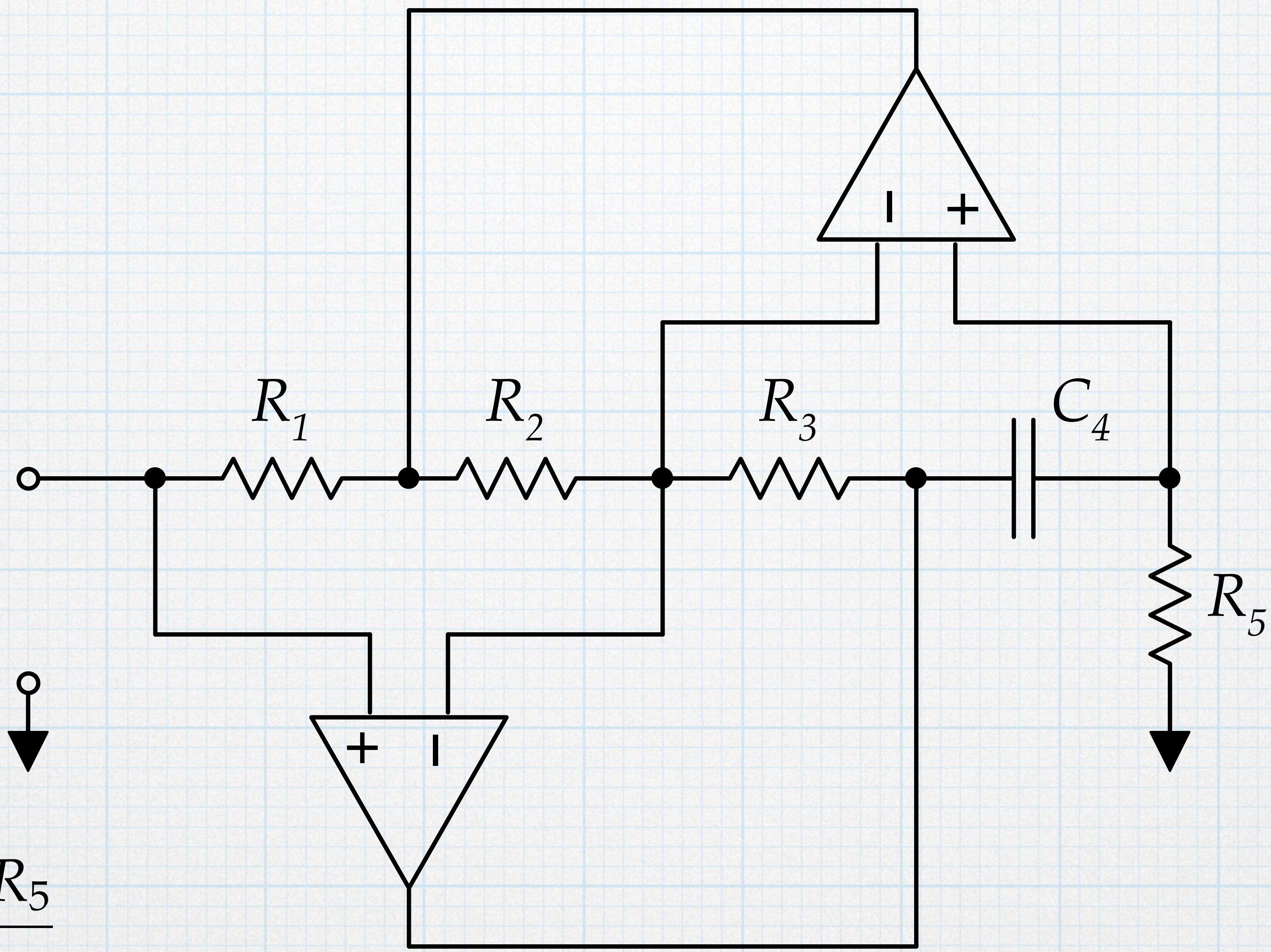
With this value of Q_P, $\omega_c = \omega_o = 2\pi(500 \text{ Hz}) = 3.14$ krad/s.

Choose $C = 1\ \mu\text{F}$ (again, somewhat arbitrarily). Then

$$L = \frac{1}{\omega_o^2 C} = \frac{1}{(3.14 \text{ krad/s})^2 (1\ \mu\text{F})} = 101 \text{ mH}$$

$$\frac{\omega_o}{Q_P} = \frac{1}{RC} \qquad R = \frac{Q_P}{\omega_o C} = \frac{0.707}{(3.14 \text{ krad/s})(1\ \mu\text{F})} = 225\ \Omega$$

Inductor simulation circuit



$Z_{in} = j\omega L$ →

$$L = C_4 \frac{R_1 R_3 R_5}{R_2}$$

$$I_5 = \frac{V_{in}}{Z_5}$$

$$I_4 = I_5 = \frac{V_{in}}{Z_5}$$

$$V_y = V_{in} + Z_4 I_4$$

$$= V_{in}\left[1 + \frac{Z_4}{Z_5}\right]$$

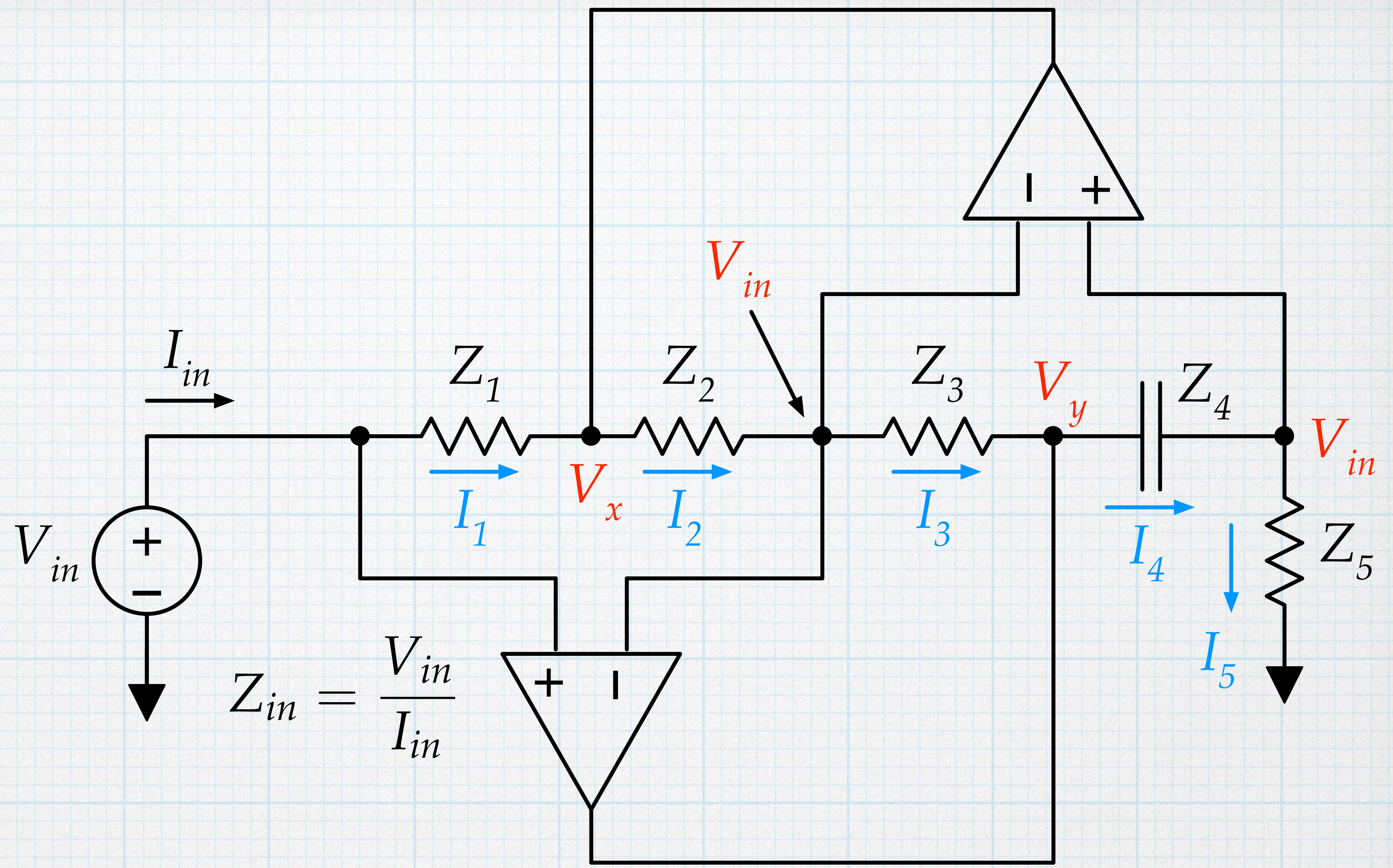
$$I_3 = \frac{V_{in} - V_y}{Z_3}$$

$$= V_{in}\frac{Z_4}{Z_3 Z_5}$$

$$I_2 = I_3$$

$$V_x = V_{in} + Z_2 I_2$$

$$= V_{in}\left[1 + \frac{Z_2 Z_4}{Z_3 Z_5}\right]$$



$$Z_{in} = \frac{V_{in}}{I_{in}}$$

$$I_{in} = I_1 = \frac{V_{in} - V_x}{Z_1}$$

$$I_{in} = V_{in}\frac{Z_2 Z_4}{Z_1 Z_3 Z_5}$$

$$Z_{in} = \frac{V_{in}}{I_{in}} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} = s\frac{R_1 R_3 R_5 C_4}{R_2} = sL$$